Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 56-2 (2023) 562–569

Robust Optimal Control With Inexact State Measurements and Adjustable Uncertainty Sets *

Yun Li *, **Neil Yorke-Smith** **, **Tamas Keviczky** *

* *Delft Center for Systems and Control, Delft University of Technology, The Netherlands (email: y.li-39@tudelft.nl; T.Keviczky@tudelft.nl)*
** *STAR Lab, Delft University of Technology, The Netherlands (email: n.yorke-smith@tudelft.nl)*

Abstract: The efficacy of robust optimal control with adjustable uncertainty sets is verified in several domains under the perfect state information setting. This paper investigates constrained robust optimal control for linear systems with linear cost functions subject to uncertain disturbances and state measurement errors that are both residing in adjustable uncertainty sets. We first show that the class of affine feedback policies of state measurements are equivalent to the class of affine feedback policies of estimated disturbances in terms of their conservativeness. Then, we formulate and solve a robust optimal control problem with adjustable uncertainty sets by considering the disturbance feedback policies. In contrast to the conventional robust optimal control, where uncertainty sets are fixed and known *a priori*, the uncertainty sets themselves are regarded as decision variables in our design. In particular, given the metrics for evaluating the optimal size/shape of the polyhedral uncertainty sets, a bilinear optimization problem is formulated to decide the optimal size/shape of uncertainty sets and a corresponding optimal control policy to robustly guarantee that the system will respect its constraints for all admissible uncertainties. In addition, we introduce a convex approximation for the proposed scheme to provide a computationally efficient inner approximation of the original problem. The proposed scheme is illustrated by numerical simulation of a building temperature control problem to demonstrate its effectiveness.

Keywords: Robust optimal control, imperfect state measurement, adjustable uncertainty set, adaptive affine policy, building temperature control

1. INTRODUCTION

Robust constrained optimal control for linear systems with additive disturbances, which usually entails optimizing control actions/policies with tightened constraints, are well investigated in both control community (Mayne et al., 2000, 2005; Chisci et al., 2001; Goulart et al., 2006; Skaf and Boyd, 2010; Langson et al., 2004; Sieber et al., 2021) and operations research (Ben-Tal and Nemirovski, 1998, 1999; Ben-Tal et al., 2004; Bertsimas and Hertog, 2022). It is commonly accepted that in order to handle larger scope of uncertainties, control policies instead of control actions should be designed. However, arbitrary/nonlinear control policies will make the resulting robust control problem computationally intractable. As a result, many methods focus on finding suboptimal but computationally tractable policies for the robust constrained control problem.

Among the existing literature, affine state feedback policies and affine disturbance feedback policies are two types of widely-adopted decision rules in robust optimal control. While it is shown in Goulart et al. (2006) that these two types of affine feedback policies are equivalent, affine feedback policies over disturbances have gained more favour due to their computational superiority: affine disturbance feedback policies yield a linear hence convex relationship between predicted states and control parameters, while affine state feedback policies give a highly nonconvex relationship between predicted states and control parameters (Goulart et al., 2006; Skaf and Boyd, 2010).

In the existing literature of developing robust control algorithms, perfect state measurement is commonly assumed. However, this assumption is not always ensured in many real-world problems. In Goulart and Kerrigan (2006); Ben-Tal et al. (2006); Richards and How (2005); Goulart and Kerrigan (2007), the robust optimal controller for linear systems with inaccurate state information is researched. Richards and How (2005) compute robust optimal control actions by solving a MPC problem with tightened constraints to counteract state estimation errors. In Ben-Tal et al. (2006), an affine decision rule over the so-called purified outputs is proposed for constrained linear systems with measurement noises. Based on a similar parameterization of decision rules as in Ben-Tal and Nemirovski (1998) but including the dynamics of a linear state observer with non-zero initial condition, an affine decision rule is designed, and its geometric and invariance

* The work was supported by the Brains4Buildings project under the Dutch grant programme for Mission-Driven Research, Development and Innovation (MOOI).

10.1016/j.ifacol.2023.10.1627

properties are investigated in Goulart and Kerrigan (2006, 2007).

It should be pointed out that all of the above-mentioned robust optimal control schemes assume that the admissible uncertainty sets are fixed and known *a priori*. However, in a so-called reserve provision problem, which prevails in operating modern energy and building systems (Fabietti et al., 2016; Mueller et al., 2019; Vrettos et al., 2016), robust optimal control with unfixed uncertainty sets need to be handled. Motivated by this problem, the constrained robust optimal control problem with adjustable uncertainty sets is formulated and studied in Zhang et al. (2017); Bitlislioğlu et al. (2017); Kim et al. (2018); Raghuraman and Koeln (2021) with the assumption of perfect state measurement. Unlike the conventional robust constrained optimal control problem, the size/shape of uncertainty sets are undetermined and treated as decision variables in the setting of adjustable uncertainty sets. The control design objective is to determine the optimal size/shape of admissible uncertainty sets and also the corresponding control inputs to robustly guarantee constraint satisfaction.

This paper provides an extension of the work in Zhang et al. (2017), which considers the robust optimal control with adjustable uncertainty set and perfect state measurements, to study the robust constrained optimal control problem in the presence of imperfect state information. Our main contributions are summarized below:

- We introduce an affine feedback policy of estimated disturbances, which yields a linear hence convex relationship between predicted system states and control parameters, and prove that this type of policy is essentially equivalent to the corresponding affine feedback policy of state measurements.
- A bilinear optimization problem is formulated to assess the optimal shape/size of the adjustable uncertainty sets and simultaneously optimize the feedback policy to robustly ensure constraint satisfaction. In addition, a convex approximation of the original bilinear optimization problem is introduced to provide a feasible inner approximation.
- We consider a building climate control problem to show how it can be formulated and solved using our proposed design framework.

The remainder of this paper is organized as follows. Section 2 describes the considered system dynamics and the robust optimal control problem. Section 3 elaborates two types of feedback control policies: affine feedback policy of state measurements and affine feedback policy of estimated disturbances, and rigorously proves their equivalence. In Section 4, restricting the uncertainty sets as polyhedra, we reformulate the original robust optimal control problem, which is semi-infinite, to yield a numerically tractable bilinear optimization problem, and also introduce a convex inner approximation for this bilinear optimization problem. We present numerical simulation results in Section 5, and conclude this paper in Section 6.

Notation: $\mathbb{R}^d$ denotes a d-dimensional real space and $\mathbb{R}^d_+$ a d-dimensional positive real space. The subscript k of a given time-dependent variable denotes k-th time step, and $k = 0$ denotes the initial time step. Uppercase letters denote matrices, and boldface lowercase letters denote stacked sequences of the given signal. $[\cdot]_i$ denotes the i-th row/element of the corresponding matrix/vector. $\mathrm{diag}(\cdot)$ defines a diagonal/block-diagonal matrix with the given matrices/vector on its diagonal. $\mathbf{I}$ denotes identity matrix with appropriate dimensions, and $\mathbf{0}$ denotes zero matrix with proper dimensions.

2. PROBLEM FORMULATION

This section introduces the formulation of the robust constrained optimal control problem. Similarly to Zhang et al. (2017), we consider the following discrete-time uncertain linear systems

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_x}$ is the unknown disturbance. For system (1), the following polytopic state-input constraints should be respected

$$Z := \{(x_{k+1}, u_k) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mid Cx_{k+1} + Du_k \leq b\} \tag{2}$$

where $C \in \mathbb{R}^{n_z \times n_x}$ and $D \in \mathbb{R}^{n_z \times n_u}$.

We assume that the disturbance signal belongs to an admissible set $\mathcal{W}$: $w_k \in \mathcal{W} \subset \mathbb{R}^{n_x}$. Unlike the problem considered in Zhang et al. (2017); Bitlislioğlu et al. (2017); Kim et al. (2018), where perfect state measurement is assumed, we consider that the system state x_k cannot be accurately measured/estimated, and the uncertain measurement error is defined as

$$e_k := \hat{x}_k - x_k, \tag{3}$$

where $\hat{x}_k$ is the measurement of x_k. Similarly, the measurement error e_k is assumed to be within an admissible uncertainty set $\mathcal{E}$: $e_k \in \mathcal{E} \subset \mathbb{R}^{n_x}$.

The reason why the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$ are called "adjustable" is because their sizes/shapes are not fixed when designing the control inputs. In contrast to conventional finite horizon robust optimal control problems that consider fixed and predetermined uncertainty sets, the parameters defining the size/shape of $\mathcal{W}$ and $\mathcal{E}$ are not fixed *a priori* and are instead decision variables in our design.

Let N be the length of the prediction horizon. Then the stacked sequences of the state $\mathbf{x}$, state measurement/estimation $\hat{\mathbf{x}}$, control input $\mathbf{u}$, disturbance $\mathbf{w}$ and measurement/estimation error $\mathbf{e}$ over N prediction steps are defined as

$$\mathbf{x} = [x_0^{\mathrm{T}}, x_1^{\mathrm{T}}, \cdots, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n_x} \tag{4a}$$
$$\hat{\mathbf{x}} = [\hat{x}_0^{\mathrm{T}}, x_i^{\mathrm{T}}, \cdots, \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n_x} \tag{4b}$$
$$\mathbf{u} = [u_0^{\mathrm{T}}, u_1^{\mathrm{T}}, \cdots, u_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_u} \tag{4c}$$
$$\mathbf{w} = [w_0^{\mathrm{T}}, w_1^{\mathrm{T}}, \cdots, w_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_x} \tag{4d}$$
$$\mathbf{e} = [e_0^{\mathrm{T}}, e_1^{\mathrm{T}}, \cdots, e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n_x}. \tag{4e}$$

Given a metric $\rho(\mathcal{W}, \mathcal{E}) := 2^{\mathbb{R}^{n_x}} \times 2^{\mathbb{R}^{n_x}} \to \mathbb{R}$ for evaluating the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$, our design objective is to find a control policy to maximize the scope of uncertainty and minimize some operational cost while ensuring constraint satisfaction for all possible uncertainty realizations $w_k \in \mathcal{W}$ and $e_k \in \mathcal{E}$. More precisely, the control objective can be formulated as

$$\min \max_{w_k\in\mathcal{W}, e_k\in\mathcal{E}} \{J(\mathbf{u},\mathbf{w},\mathbf{e})\} - \lambda\cdot\rho(\mathcal{W},\mathcal{E}) \tag{5a}$$
$$\text{s.t. (1) and } (x_{k+1},u_k)\in Z \tag{5b}$$
$$\forall w_k\in\mathcal{W},\ k=0,1,\cdots,N-1 \tag{5c}$$
$$\forall e_k\in\mathcal{E},\ k=0,1,\cdots,N \tag{5d}$$

where $\lambda \geq 0$ is a user-defined weighting parameter, $J(\mathbf{u},\mathbf{w},\mathbf{e})$ is the nominal operational cost function, and the minimization is performed over control inputs and the parameters defining the shape/size of the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$. A common choice for $J(\mathbf{u},\mathbf{w},\mathbf{e})$ is

$$J(\mathbf{u},\mathbf{w},\mathbf{e}) := \sum_{k=0}^{N-1} l_k(\phi_{k+1}(\mathbf{u},\mathbf{w},\mathbf{e}),u_k) \tag{6}$$

where $l_k(\phi_{k+1}(\mathbf{u},\mathbf{w},\mathbf{e}),u_k)$ is the k-th stage cost function, and $\phi_k(\mathbf{u},\mathbf{w},\mathbf{e})$ represents the system states at k-th time step driven by $(\mathbf{u},\mathbf{w},\mathbf{e})$. As in Goulart et al. (2006); Zhang et al. (2017), stage cost functions are restricted to linear functions. Possible choices for selecting $\rho(\cdot,\cdot)$ can be found in Zhang et al. (2017).

3. STATE AND DISTURBANCE FEEDBACK POLICIES AND THEIR EQUIVALENCE

In this section, two types of control policies: *state feedback policy* and *disturbance feedback policy*, will be discussed. We will show that, while the disturbance feedback policy is more computationally superior, these two types of control policies are equivalent even in the presence of inaccurate state information.

3.1 State Feedback Policy

One natural choice for the control policy is to parameterize the control input with system states in an affine structure as in Goulart et al. (2006) and Skaf and Boyd (2010). Since imperfect state measurement/estimation is assumed in our design setting, the state feedback policy is consequently constructed as

$$u_k^{\text{sf}} = \sum_{i=0}^{k} L_{k,i}\hat{x}_i + g_k \tag{7}$$

where $L_{k,i}\in\mathbb{R}^{n_u\times n_x}$ and $g_k\in\mathbb{R}^{n_u}$ are control parameters to be optimized. Since the controller should be non-anticipative, u_k^{sf} is only dependent on measurements x_t for $t\leq k$. The input sequence $\mathbf{u}$ can then be compactly expressed as

$$\mathbf{u}^{\mathbf{sf}} = \mathbf{L}\hat{\mathbf{x}} + \mathbf{g} \tag{8}$$

where $\mathbf{L}\in\mathbb{R}^{Nn_u\times(N+1)n_x}$ and $\mathbf{g}\in\mathbb{R}^{Nn_u}$ are defined as

$$\mathbf{L} = \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}. \tag{9}$$

For a given initial state x, the state feedback policy $(\mathbf{L},\mathbf{g})$ is admissible if the control policy (8) guarantees that the system constraints (2) are satisfied for all possible uncertainties $w_k\in\mathcal{W}$ and $e_k\in\mathcal{E}$ over the N-step prediction horizon. More precisely, the set of feasible state feedback policies is defined as

$$\Pi^{\text{sf}}(x) := \left\{ (\mathbf{L},\mathbf{g}) \;\middle|\; \begin{array}{l} x_{k+1} = Ax_k + Bu_k + w_k \\ u_k = \sum_{i=0}^{k} L_{k,i}\hat{x}_i + g_k \\ (\mathbf{L},\mathbf{g}) \text{ satisfy (9)} \\ x_0 = x \\ (x_{k+1},u_k)\in Z \\ \forall\ w_k\in\mathcal{W},\ \forall\ e_k\in\mathcal{E} \\ \forall\ k\in[0,1,\cdots,N-1] \end{array} \right\}. \tag{10}$$

Correspondingly, the set of initial states that ensures the existence of a feasible state feedback policy is

$$X^{\text{sf}} := \{x\in\mathbb{R}^{n_x} \,|\, \Pi^{\text{sf}}(x)\neq\emptyset\}. \tag{11}$$

As mentioned in Goulart et al. (2006), one critical limitation of the state feedback control policy is that the set of admissible state feedback policies $\Pi^{\text{sf}}(x)$ is non-convex. Hence, in order to optimize the control parameters $(\mathbf{L},\mathbf{g})$, a non-convex optimization problem needs to be solved, or some nonlinear reformulations need to be performed.

3.2 Disturbance Feedback Policy

An alternative parameterization of the control policy is to design the input as an affine function of prior disturbances, i.e., $u_k^{\text{df}} = \sum_{i=0}^{k-1} M_{k,i}w_i + v_k$. Since the disturbance signal w_k is only available after $(k+1)$-st time step, the control input u_k^{df} is consequently constructed by w_t for $t\leq k-1$. In many cases the unknown external disturbance w_k is not directly measurable, and is rather computed via $w_k = x_{k+1} - Ax_k - Bu_k$ (Goulart et al., 2006). In our work, an estimation of the disturbance from imperfect state measurements is computed as

$$\hat{w}_k = \hat{x}_{k+1} - A\hat{x}_k - Bu_k \tag{12}$$

and accordingly, the control policy u_k^{df} becomes

$$u_k^{\text{df}} = \sum_{i=0}^{k-1} M_{k,i}\hat{w}_i + v_k. \tag{13}$$

The input sequence over the N prediction steps $\mathbf{u}^{\mathbf{df}}$ can then be compactly represented as

$$\mathbf{u}^{\mathbf{df}} = \mathbf{M}\hat{\mathbf{w}} + \mathbf{v} \tag{14}$$

where $\hat{\mathbf{w}}\in\mathbb{R}^{Nn_x}$, $\mathbf{M}\in\mathbb{R}^{Nn_u\times Nn_x}$ and $\mathbf{v}\in\mathbb{R}^{Nn_u}$ are defined as $\hat{\mathbf{w}} = [\hat{w}_0^{\mathrm{T}},\hat{w}_1^{\mathrm{T}},\cdots,\hat{w}_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}. \tag{15}$$

Similarly to (10) and (11), the set of feasible disturbance feedback policies $(\mathbf{M},\mathbf{v})$ and the corresponding set of feasible initial states are defined as

$$\Pi^{\text{df}}(x) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{l} x_{k+1} = Ax_k + Bu_k + w_k \\ u_k = \sum_{i=0}^{k-1} M_{k,i}\hat{w}_i + v_k \\ \hat{w}_k = \hat{x}_{k+1} - A\hat{x}_k - Bu_k \\ (\mathbf{M},\mathbf{v}) \text{ satisfy (15)} \\ x_0 = x \\ (x_{k+1},u_k)\in Z \\ \forall w_k\in\mathcal{W},\ \forall e_k\in\mathcal{E} \\ \forall k\in[0,1,\cdots,N-1] \end{array} \right\} \tag{16}$$

$$X^{\mathrm{df}} := \{x \in \mathbb{R}^{n_x} \big| \Pi^{\mathrm{df}}(x) \neq \emptyset\}. \tag{17}$$

One critical advantage of the disturbance feedback policy (14) over the state feedback policy (8) is that the predicted state sequence $\mathbf{x}$ is an affine function of the control parameters $(\mathbf{M}, \mathbf{v})$.

3.3 Equivalence Between State and Disturbance Feedback Policies

Regarding the state feedback policy (7) and the disturbance feedback policy (13), one natural question to ask is whether one policy is more or less conservative than the other. It has been shown in Goulart et al. (2006) that assuming perfect state measurements these two types of control policies are equivalent. In the following, we will show that the two types of control policies are equivalent even under imperfect state measurement setting.

Theorem 1. The admissible sets of initial states for both state feedback and disturbance feedback policies in (11) and (17), respectively, are identical, namely $X^{\mathrm{df}} = X^{\mathrm{sf}}$. In addition, for any feasible state feedback policy $(\mathbf{L}, \mathbf{g})$, an equivalent feasible disturbance feedback policy $(\mathbf{M}, \mathbf{v})$ can be found to yield the same state and input sequences for any allowable uncertainty sequences $\mathbf{w}$ and $\mathbf{e}$, and vice-versa.

Proof: For system (1), the state sequence over N-step prediction horizon $\mathbf{x}$ can be compactly represented as

$$\mathbf{x} = \mathbf{A}x_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} \\ A \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \tag{19}$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathbf{I} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1} & A^{N-2} & \cdots & \mathbf{I} \end{bmatrix}. \tag{20}$$

In addition, according to (1), (3) and (12), we have

$$\begin{aligned} \hat{w}_k &= x_{k+1} + e_{k+1} - A(x_k + e_k) - Bu_k & \text{(21a)} \\ &= w_k + e_{k+1} - Ae_k & \text{(21b)} \end{aligned}$$

Defining $\tilde{w}_k := \hat{w}_k - w_k$ and considering (21) yields

$$\tilde{\mathbf{w}} = \mathbf{H}\mathbf{e} \tag{22}$$

where $\tilde{\mathbf{w}} \in \mathbb{R}^{Nn_x}$ and $\mathbf{H} \in \mathbb{R}^{Nn_x \times (N+1)n_x}$ are

$$\tilde{\mathbf{w}} := \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_{N-1} \end{bmatrix}, \quad \mathbf{H} := \begin{bmatrix} -A & \mathbf{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -A & \mathbf{I} \end{bmatrix}. \tag{23}$$

$X^{\mathrm{sf}} \subseteq X^{\mathrm{df}}$: From the definition of X^{sf}, given $x_0 \in X^{\mathrm{sf}}$, there exists at least one state feedback control policy $(\mathbf{L}, \mathbf{g})$ satisfying the constraints in (10). For admissible sequences $\mathbf{w}$ and $\mathbf{e}$, the input sequence generated from the state feedback policy and the corresponding state sequence are

$$\begin{aligned} \mathbf{u}^{\mathbf{sf}} &= \mathbf{L}\hat{\mathbf{x}} + \mathbf{g} & \text{(24a)} \\ \mathbf{x} &= \mathbf{A}x_0 + \mathbf{B}(\mathbf{L}\hat{\mathbf{x}} + \mathbf{g}) + \mathbf{E}\mathbf{w}. & \text{(24b)} \end{aligned}$$

Based on the definition of $\mathbf{e}$ in (3), (24b) can be rewritten as

$$\mathbf{x} = \mathbf{A}\hat{x}_0 - \mathbf{A}e_0 + \mathbf{B}(\mathbf{L}\mathbf{x} + \mathbf{g}) + \mathbf{B}\mathbf{L}\mathbf{e} + \mathbf{E}\mathbf{w} \tag{25}$$

which further gives

$$\mathbf{x} = (\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}\hat{x}_0 - \mathbf{A}e_0 + \mathbf{B}\mathbf{g} + \mathbf{B}\mathbf{L}\mathbf{e} + \mathbf{E}\mathbf{w}). \tag{26}$$

Substituting (26) into (24a) yields

$$\begin{aligned} \mathbf{u}^{\mathbf{sf}} &= \mathbf{L}\mathbf{x} + \mathbf{g} + \mathbf{L}\mathbf{e} & \text{(27a)} \\ &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}\hat{x}_0 + \mathbf{B}\mathbf{g}) + \mathbf{g} \\ &\quad + \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{L}\mathbf{e} - \mathbf{A}e_0) + \mathbf{L}\mathbf{e}. & \text{(27b)} \end{aligned}$$

Notice that matrix $(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}$ is always well-defined since $\mathbf{B}\mathbf{L}$ is strictly lower triangular. Defining $\bar{\mathbf{A}} = [\mathbf{A}, \mathbf{0}, \cdots, \mathbf{0}] \in \mathbb{R}^{(N+1)n_x \times (N+1)n_x}$ leads to

$$\mathbf{A}e_0 = \bar{\mathbf{A}}\mathbf{e}. \tag{28}$$

In order to find a disturbance feedback policy $\mathbf{u}^{\mathbf{df}} = \mathbf{M}\hat{\mathbf{w}} + \mathbf{v} = \mathbf{M}\mathbf{w} + \mathbf{v} + \mathbf{M}\tilde{\mathbf{w}} = \mathbf{M}\mathbf{w} + \mathbf{v} + \mathbf{M}\mathbf{H}\mathbf{e}$ that is equivalent to (27), the following should hold

$$\begin{aligned} \mathbf{M} &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E} & \text{(29a)} \\ \mathbf{v} &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}\hat{x}_0 + \mathbf{B}\mathbf{g}) + \mathbf{g} & \text{(29b)} \\ \mathbf{M}\mathbf{H} &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{L} - \bar{\mathbf{A}}) + \mathbf{L}. & \text{(29c)} \end{aligned}$$

Considering (29a) and (29c), the following relationship should be satisfied to ensure that the above equations hold

$$\underbrace{\mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{H}}_{\text{LHS}} = \underbrace{\mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{L} - \bar{\mathbf{A}}) + \mathbf{L}}_{\text{RHS}}. \tag{30}$$

The RHS can be rewritten as

$$\begin{aligned} \text{RHS} &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{L} - \bar{\mathbf{A}}) + \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{I} - \mathbf{B}\mathbf{L}) \\ &= \mathbf{L}(\mathbf{I} - \mathbf{B}\mathbf{L})^{-1}(\mathbf{I} - \bar{\mathbf{A}}). \end{aligned}$$

Recalling the definition of $\mathbf{E}$, $\mathbf{H}$ and $\bar{\mathbf{A}}$, it is easy to verify that $\mathbf{E}\mathbf{H} = \mathbf{I} - \bar{\mathbf{A}}$, which justifies (30). Further, it can be readily checked that the $(\mathbf{M}, \mathbf{v})$ defined in (29a) and (29b) also satisfies the structure in (15) and will generate the same state sequence as with the corresponding state feedback policy. Hence, $(\mathbf{M}, \mathbf{v}) \in \Pi^{\mathrm{df}}(x_0)$ and $x_0 \in X^{\mathrm{sf}} \Rightarrow x_0 \in X^{\mathrm{df}}$.

$X^{\mathrm{df}} \subseteq X^{\mathrm{sf}}$: According to the definition of X^{df}, there exists a disturbance feedback policy $(\mathbf{M}, \mathbf{v})$ such that the constraints in (16) are satisfied. Given a sequence of disturbances $\mathbf{w}$, and a sequence of measurement errors $\mathbf{e}$, the corresponding input and state sequences are

$$\begin{aligned} \mathbf{u}^{\mathbf{df}} &= \mathbf{M}\hat{\mathbf{w}} + \mathbf{v} = \mathbf{M}\mathbf{w} + \mathbf{v} + \mathbf{M}\mathbf{H}\mathbf{e} & \text{(31a)} \\ \mathbf{x} &= \mathbf{A}x_0 + \mathbf{B}\mathbf{u}^{\mathbf{df}} + \mathbf{E}\mathbf{w}. & \text{(31b)} \end{aligned}$$

Since $\mathbf{E}$ is full column rank, there exists a matrix $\mathbf{E}^* \in \mathbb{R}^{Nn_x \times (N+1)n_x}$ such that $\mathbf{E}^*\mathbf{E} = \mathbf{I}$. Then, based on (31b), we have

$$\mathbf{w} = \mathbf{E}^* \left(\mathbf{x} - \mathbf{A}x_0 - \mathbf{B}\mathbf{u}^{\mathbf{df}} \right). \tag{32}$$

Substituting (32) into (31a) leads to

$$\begin{aligned} \mathbf{u}^{\mathbf{df}} &= \left(\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B}\right)^{-1}\left(\mathbf{M}\mathbf{E}^*(\mathbf{x} - \mathbf{A}x_0) + \mathbf{v} + \mathbf{M}\mathbf{H}\mathbf{e}\right) \\ &= (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^*\mathbf{x} + (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}(\mathbf{v} \\ &\quad - \mathbf{M}\mathbf{E}^*\mathbf{A}\hat{x}_0) + (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}\mathbf{M}(\mathbf{E}^*\bar{\mathbf{A}} + \mathbf{H})\mathbf{e} \end{aligned} \tag{33}$$

where $(\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}$ is always well-defined since $\mathbf{M}\mathbf{E}^*\mathbf{B}$ is strictly lower triangular. In order to find an equivalent state feedback policy $\mathbf{u^{sf}} = \mathbf{L}\hat{\mathbf{x}} + \mathbf{g} = \mathbf{L}\mathbf{x} + \mathbf{g} + \mathbf{L}\mathbf{e}$, the following should hold

$$\mathbf{L} = (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^* \quad (34a)$$

$$\mathbf{g} = (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}\left(-\mathbf{M}\mathbf{E}^*\mathbf{A}\hat{x}_0 + \mathbf{v}\right) \quad (34b)$$

$$\mathbf{L} = (\mathbf{I} + \mathbf{M}\mathbf{E}^*\mathbf{B})^{-1}\mathbf{M}(\mathbf{E}^*\bar{\mathbf{A}} + \mathbf{H}). \quad (34c)$$

By checking the structure of $\mathbf{H}$ and $\mathbf{E}$, it can be verified that $\mathbf{H}\mathbf{E} = \mathbf{I}$, namely $\mathbf{E}^* = \mathbf{H}$, and $\mathbf{H}\bar{\mathbf{A}} = \mathbf{0}$, which justify (34a) and (34c). In addition, the state feedback policy $(\mathbf{L}, \mathbf{g})$ defined in (34a) and (34b) also satisfies (10), and will generate the same sequence of states as with the corresponding disturbance feedback policy. As a result, for every $x_0 \in X^{\mathrm{df}}$, it can be concluded that $x_0 \in X^{\mathrm{sf}}$. In other words, $X^{\mathrm{df}} \subseteq X^{\mathrm{sf}}$. This completes the proof. □

Remark 1: Unlike the control input parameterization adopted in Goulart and Kerrigan (2007) and Ben-Tal et al. (2006), where so-called purified output signals are used and a linear observer is designed, we parameterize the control input as an affine function of estimated disturbances that are computed from state measurements/estimations, which is similar to the parameterization considered in Goulart et al. (2006) and Zhang et al. (2017) but with noisy state information. If we would further assume that system states are fully measurable and used directly for constructing the control policy in (7), namely the output measurements $y = \hat{x}$, the scheme designed in Goulart and Kerrigan (2007) subsumes our proposed scheme. However, it is worth emphasizing that, for applying the schemes in Goulart and Kerrigan (2007) and Ben-Tal et al. (2006), a linear state observer with predefined structures needs to be designed for obtaining the purified output signal. On the contrary, our proposed scheme imposes no restrictions on the approaches for obtaining the state estimate $\hat{x}$. For example, moving horizon estimation (MHE) (Rao et al., 2003), which is effective in constrained state estimation, can be combined with our scheme directly but is not compatible with the schemes proposed in Goulart and Kerrigan (2007) and Ben-Tal et al. (2006). In addition, we show in Appendix A that if the state estimate is assumed to be obtained via the linear observer considered in Goulart and Kerrigan (2007), the control policy considered in this work also subsumes the policy considered in Goulart and Kerrigan (2007) under some mild conditions.

4. ROBUST CONTROL WITH ADJUSTABLE UNCERTAINTY SETS

In this section, the robust constrained optimal control problem with imperfect state measurements and adjustable uncertainty sets formulated in Section 2 is solved using the duality of linear optimization. Since we have proven in Section 3 that the disturbance feedback policy (14) is equivalent to the state feedback policy (8), in the following we will only investigate the robust optimal control problem formulated in (5) with the disturbance feedback policy because of its computational superiority.

In our design, the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$ are restricted to be compact (closed and bounded) polyhedra and contain the origin in their interior, which are defined as

$$\mathcal{W} := \{w \mid Fw \le f\} \quad (35a)$$

$$\mathcal{E} := \{e \mid Ye \le y\} \quad (35b)$$

where $(F, f) \in \mathbb{F} \subseteq \mathbb{R}^{n_f \times n_x} \times \mathbb{R}^{n_f}$ and $(Y, y) \in \mathbb{Y} \subseteq \mathbb{R}^{n_y \times n_x} \times \mathbb{R}^{n_y}$ are unknown or partially unknown parameters to be determined. To ensure that the resulting optimization problem for determining the optimal uncertainty sets and the corresponding control policy only contain bilinear constraints, the admissible sets of control parameters $\mathbb{F}$ and $\mathbb{Y}$ are also restricted to be polyhedra.

By defining the N-fold Cartesian product of $\mathcal{W}$ and $\mathcal{E}$ as $\mathcal{W}^N := \mathcal{W} \times \mathcal{W} \cdots \mathcal{W}$ and $\mathcal{E}^N := \mathcal{E} \times \mathcal{E} \cdots \mathcal{E}$, respectively, the admissible sets of the uncertain sequences $\mathbf{w}$ and $\mathbf{e}$ can be expressed as

$$\mathcal{W}^N := \{\mathbf{w} \mid \mathbf{F}\mathbf{w} \le \mathbf{f}\} \quad (36)$$

where $\mathbf{F} = \mathrm{diag}(F, \cdots, F) \in \mathbb{R}^{Nn_f \times Nn_x}$ and $\mathbf{f} = [f^{\mathrm{T}}, \cdots, f^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_f}$;

$$\mathcal{E}^N := \{\mathbf{e} \mid \mathbf{Y}\mathbf{e} \le \mathbf{y}\} \quad (37)$$

where $\mathbf{Y} = \mathrm{diag}(Y, \cdots, Y) \in \mathbb{R}^{Nn_y \times Nn_x}$ and $\mathbf{y} = [y^{\mathrm{T}}, \cdots, y^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_y}$.

Recall that the nominal cost function in (5) is restricted to be linear. Then considering (2), (14) and (18), the optimal control problem (5) can be formulated as

$$\min_{\mathbf{M},\mathbf{v},F,f,Y,y} \Big\{ \max_{\mathbf{w}\in\mathcal{W}^N, \mathbf{e}\in\mathcal{E}^N} \{\mathbf{p}^{\mathrm{T}}\mathbf{u^{df}} + \mathbf{q}^{\mathrm{T}}\mathbf{w} + \mathbf{o}^{\mathrm{T}}\mathbf{e}\} - \lambda \cdot \rho(\mathcal{W}, \mathcal{E}) \Big\} \quad (38a)$$

$$\text{s.t.} \quad \mathbf{u^{df}} = \mathbf{M}\hat{\mathbf{w}} + \mathbf{v}, \ (\mathbf{M}, \mathbf{v}) \text{ satisfies (15)} \quad (38b)$$

$$\mathbf{C}\mathbf{u^{df}} + \mathbf{D}\mathbf{w} + \mathbf{G}\mathbf{e} \le \mathbf{d} \quad (38c)$$

$$\forall \mathbf{w} \in \mathcal{W}^N, \ \forall \mathbf{e} \in \mathcal{E}^N \quad (38d)$$

where $\mathbf{p}$, $\mathbf{q}$, $\mathbf{o}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{d}$ are constructed from system parameters and exogenous information in the objective function.

The problem formulation (38) requires the satisfaction of infinitely many constraints and is a semi-infinite optimization problem, which is computationally intractable for numerical solvers. The universal quantifier in (38d) can be equivalently replaced by guaranteeing constraint satisfaction for the following worst-case scenario

$$\min \ \tau - \lambda \cdot \rho(\mathcal{W}, \mathcal{E}) \quad (39a)$$

$$\text{s.t.} \ \max_{\mathbf{w},\mathbf{e}} \left\{(\mathbf{p}^{\mathrm{T}}\mathbf{M} + \mathbf{q}^{\mathrm{T}})\mathbf{w} + (\mathbf{p}^{\mathrm{T}}\mathbf{M}\mathbf{H} + \mathbf{o}^{\mathrm{T}})\mathbf{e}\right\} + \mathbf{p}^{\mathrm{T}}\mathbf{v} \le \tau \quad (39b)$$

$$\max_{\mathbf{w},\mathbf{e}} \{(\mathbf{C}\mathbf{M} + \mathbf{D})\mathbf{w} + (\mathbf{C}\mathbf{M}\mathbf{H} + \mathbf{G})\mathbf{e}\} + \mathbf{C}\mathbf{v} \le \mathbf{d} \quad (39c)$$

$$\mathbf{F}\mathbf{w} \le \mathbf{f}, \ \mathbf{Y}\mathbf{e} \le \mathbf{y} \quad (39d)$$

$$(F, f) \in \mathbb{F}, \ (Y, y) \in \mathbb{Y}, \ (\mathbf{M}, \mathbf{v}) \text{ satisfies (15)} \quad (39e)$$

where the max operator in (39c) represents row-wise maximization. Notice that in (39) the objective function and constraints are linear w.r.t. $\mathbf{w}$ and $\mathbf{e}$. Following a similar line as in Zhang et al. (2017) and applying the duality of LP, problem (39) can then be equivalently reformulated as

$$\min \tau - \lambda \cdot \rho(\mathcal{W}, \mathcal{E}) \tag{40a}$$
$$\text{s.t. } \mathbf{f}^{\mathrm{T}}\mu_1 + \mathbf{y}^{\mathrm{T}}\mu_2 + \mathbf{p}^{\mathrm{T}}\mathbf{v} \le \tau \tag{40b}$$
$$\mathbf{F}^{\mathrm{T}}\mu_1 = \mathbf{M}^{\mathrm{T}}\mathbf{p} + \mathbf{q} \tag{40c}$$
$$\mathbf{Y}^{\mathrm{T}}\mu_2 = \mathbf{H}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{p} + \mathbf{o} \tag{40d}$$
$$\mu_1 \ge 0,\ \mu_2 \ge 0 \tag{40e}$$
$$\mathbf{f}^{\mathrm{T}}\eta_1^i + \mathbf{y}^{\mathrm{T}}\eta_2^i + [\mathbf{C}]_i^{\mathrm{T}}\mathbf{v} \le \mathbf{d}_i \tag{40f}$$
$$\mathbf{F}^{\mathrm{T}}\eta_1^i = [\mathbf{CM} + \mathbf{D}]_i \tag{40g}$$
$$\mathbf{Y}^{\mathrm{T}}\eta_2^i = [\mathbf{CMH} + \mathbf{G}]_i \tag{40h}$$
$$(\mathbf{M}, \mathbf{v}) \text{ satisfies (15)} \tag{40i}$$
$$\eta_1^i \ge 0,\ \eta_2^i \ge 0,\ i = 1, 2, \cdots, Nn_z \tag{40j}$$
$$(F, f) \in \mathbb{F},\ (Y, y) \in \mathbb{Y} \tag{40k}$$

where $(\mathbf{M}, \mathbf{v}, F, f, Y, y, \mu_1, \mu_2, \eta_1^i, \eta_2^i, \tau)$ are decision variables. By solving this optimization problem, the optimal solution $(\mathbf{M}^*, \mathbf{v}^*, F^*, f^*, Y^*, y^*)$ will characterize the admissible uncertainty sets $\mathcal{W}^N$ and $\mathcal{E}^N$ such that for all possible $\mathbf{w} \in \mathcal{W}^N$ and $\mathbf{e} \in \mathcal{E}^N$ there exists a disturbance feedback control policy $(\mathbf{M}^*, \mathbf{v}^*)$ to robustly guarantee constraint satisfaction for the system over the N-step prediction horizon.

Remark 2: The reason why we restrict the nominal operational cost function to be linear and define the feasible sets of all decision variables via linear constraints is to ensure that the resulting optimization only contains one type of non-convexity: bilinear terms $(\mathbf{F}^{\mathrm{T}}\mu_1, \mathbf{Y}^{\mathrm{T}}\mu_2, \mathbf{F}^{\mathrm{T}}\eta_1^i, \mathbf{Y}^{\mathrm{T}}\eta_2^i)$ in the constraints (40b) – (40k). Assuming that the objective function $\rho(\mathcal{W}, \mathcal{E})$ is a linear/bilinear function of (F, f, Y, y) will then ensure that (40) is a bilinear optimization problem, which can be handled by several off-the-shelf solvers, such as `Gurobi`, `SCIP` and `Ipopt`.

Remark 3: For the optimization problem (40), we can eliminate the bilinearities in (40b) – (40k) such that the resulting optimization problem only contains linear hence convex constraints to provide an inner approximation of the original problem (39). This can be achieved by making two modifications to the original problem: 1) considering control actions $\mathbf{u}^{\mathbf{df}} = \mathbf{v}$ instead of control policies $\mathbf{u} = \mathbf{M}\hat{\mathbf{w}} + \mathbf{v}$; and 2) applying the uncertainty set approximation via primitive sets introduced in Zhang et al. (2017) for the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$. Then, it can be verified that the dual problem of the modified optimization problem only contains linear hence convex constraints. Hence, the computational tractability of the resulting optimization problem will be improved, and any feasible solution to this simplified problem is also feasible for the original problem with affine control policies.

5. SIMULATION RESULTS

The value of the methodology developed in Section 4 is now illustrated through a case study of building climate control. We consider a single-zone building model that is identified via the resistor-capacitor (RC)-network approach with a 2R2C structure (Bacher and Madsen, 2011). The mathematical expression of the indoor temperature dynamics is

$$x_{k+1} = Ax_k + Bu_k + w_k + Rd_k, \tag{41}$$

where the states $x_k = [x_{k,1}, x_{k,2}] \in \mathbb{R}^2$ represent the average indoor temperature $x_{k,1}$ and the average building envelope temperature $x_{k,2}$, $u_k \in \mathbb{R}$ denotes the controllable heating/cooling power, $w_k \in \mathbb{R}^2$ the scaled heating/cooling power generated by indoor appliances or flexible power adjustment for providing demand-side service to power grid, and $d_k \in \mathbb{R}^2$ the boundary conditions: outdoor temperature and solar radiation. While the building thermal dynamics in (41) is not in the same format as the system in (1), the proposed approach is still directly applicable since all system constraints considered are linear.

It should be mentioned that accurately measuring or estimating the average temperatures for both the indoor air and building's envelope is difficult since the measurement/estimation of the temperature variables is influenced by several factors, such as sensor locations, humidity, radiation, etc, and also the non-homogeneity of construction materials. In addition, the indoor climate is subject to some uncertain internal heat gains, such as heat flux from appliance and occupancy behaviors. In order to efficiently operate the building temperature control system and maintain comfortable indoor conditions, the uncertainties in both the temperature measurement/estimation and the internal heat gains should be properly considered. In the following, we will apply the proposed scheme to analyze the maximal size of permissible uncertainties in both indoor temperature measurements/estimations as well as internal heat gains and design a corresponding heating/cooling strategy to achieve robust comfort constraint satisfaction.

In our simulation example, comfort constraints are given as $19°\mathrm{C} \le x_{k,1} \le 24°\mathrm{C}$. Heating power constraints are $-2000\mathrm{W} \le u_k \le 2000\mathrm{W}$. The admissible uncertainty sets for w_k and e_k are defined as

$$-\bar{w} \le w_k \le \bar{w},\ -\bar{e} \le e_k \le \bar{e}$$

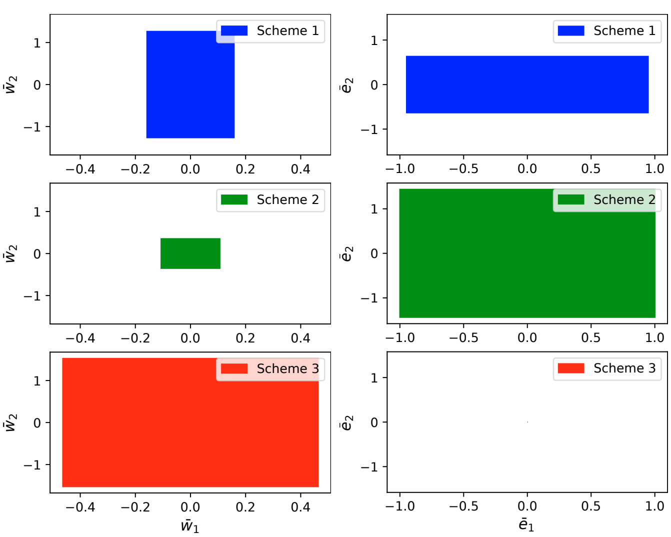
where $\bar{w} = [\bar{w}_1, \bar{w}_2] \in \mathbb{R}_+^2$ and $\bar{e} = [\bar{e}_1, \bar{e}_2] \in \mathbb{R}_+^2$ are decision variables to be determined. Model parameters A, B and R are identified according to the data provided in Rouchier (2022) using `Scipy`. Optimization problems are modeled using `Pyomo` and solved via `Gurobi 9.5.1` solver. Scripts for reproducing our simulation results are available in `https://github.com/li-yun/optimal_control_inexact_measurement`.

Notice that the work of Goulart and Kerrigan (2007) requires that the uncertainty sets $\mathcal{W}$ and $\mathcal{E}$ are fixed. However, this paper aims at exploring the problem of robust optimal control for adjustable uncertainty sets, thus a comparable design from Zhang et al. (2017) is considered in our simulation for comparison. We implement the following three schemes:

- Scheme 1: the robust control formulation defined in (40).
- Scheme 2: the convex approximation of (40) introduced in *Remark 3*.
- Scheme 3: the approach proposed in Zhang et al. (2017), where the state measurement error is not considered.

The objective function for all considered schemes is set as $\rho(\mathcal{W}, \mathcal{E}) = \bar{w}_1\bar{w}_2\bar{e}_1\bar{e}_2$, which is to maximize the product of the area of the rectangles defined by $\bar{w}$ and $\bar{e}$. The estimated initial states is $x_0 = [20°C, 25°C]$, and the prediction horizon is $N = 8$. The admissible uncertainty sets $\mathcal{W}$ and $\mathcal{E}$ computed via the three different schemes are depicted in Fig. 1. It can be seen that Scheme 3 obtains the largest admissible uncertainty set for disturbance w_k since no state measurement error e_k is considered. Compared with Scheme 2, which is a convex approximation of Scheme 1, Scheme 1 gives a larger value of objective function, namely a larger value of the product of the area of two uncertainty sets.

Remark 4: The reasons for choosing the objective function $\rho = \bar{w}_1\bar{w}_2\bar{e}_1\bar{e}_2$ are to avoid $\bar{w}_i^* = 0$ or $\bar{e}_i^* = 0$, which is undesirable since in that case no uncertainty can be tolerated for w_i or e_i, and to balance the size of two uncertainty sets. After obtaining the values of $\bar{w}_i^*$ and $\bar{e}_i^*$, we can then actively determine the accuracy of state measurement/estimation, e.g., by choosing appropriate sensors or estimation algorithms, and select the set of the unmodeled thermal disturbance w, e.g., by adjusting the model accuracy. As long as the uncertainties belong to the uncertainty sets defined by $(\bar{w}^*, \bar{e}^*)$, system constraints can be guaranteed by applying the corresponding control policy $(\mathbf{M}, \mathbf{v})$. Since the uncertainty sets are adjustable and can be predefined before the uncertainties are revealed, constraint satisfaction is guaranteed *a priori*.



(a) admissible set of $\mathcal{W}$ (b) admissible set of $\mathcal{E}$

Fig. 1. Admissible uncertainty sets for the considered three design schemes.

Based on the admissible uncertainty sets computed via Scheme 1, we randomly generate 50,000 feasible sequences of $\mathbf{w}$ and $\mathbf{e}$, respectively, and then implement the optimal control policies $(\mathbf{M}^*, \mathbf{v}^*)$ solved by different schemes. The indoor temperature envelopes with different control schemes are depicted in Fig. 2 (dotted lines indicate comfort constraints). Observe while Scheme 3 can theoretically tolerate the largest admissible uncertainty set $\mathcal{W}$ by ignoring state measurement errors, indoor comfort constraints can be violated by applying Scheme 3 even in the presence of a smaller uncertainty $\mathbf{w}$ but with state measurement errors. This observation justifies the necessity of considering measurement errors explicitly. Compared with Scheme 1, Scheme 2 is more computationally efficient since only control actions are considered and bilinearities in constraints are removed. However, indoor constraints will be violated in the presence of greater uncertainty in w_k.

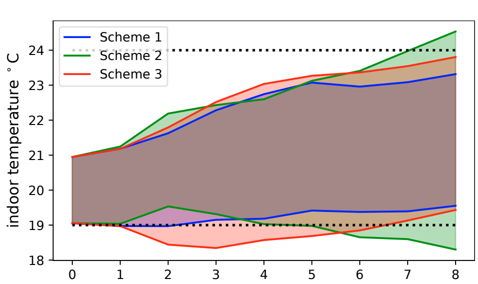


Fig. 2. Indoor temperature envelopes for the considered three design schemes.

6. CONCLUSION

We studied the robust optimal control design problem for linear systems with linear cost functions and adjustable uncertainty sets in the presence of imperfect state measurements. We have shown that the class of affine policies of state measurements is equivalent to the class of affine policies of estimated disturbances. Instead of assuming fixed uncertainty sets as within the conventional robust optimal control framework, we consider optimizing the size/shape of polyhedral uncertainty sets of disturbances and state measurement errors. By considering polyhedral uncertainty sets, the proposed approach is able to determine the optimal size/shape of the uncertainty sets and also a feasible control policy to robustly guarantee constraint satisfaction by solving a bilinear optimization problem. The applicability of the proposed approach is exemplified by implementing the designed scheme on a building indoor temperature control problem.

Future extensions may include seeking computationally efficient formulations and numerical solutions for the proposed design without increasing conservativeness, and investigating recursive feasibility/stability issues for the proposed approach.

REFERENCES

Bacher, P. and Madsen, H. (2011). Identifying suitable models for the heat dynamics of buildings. *Energy and Buildings*, 43(7), 1511–1522.

Ben-Tal, A., Boyd, S., and Nemirovski, A. (2006). Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems. *Mathematical Programming*, 107(1), 63–89.

Ben-Tal, A., Goryashko, A., Guslitzer, E., and Nemirovski, A. (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2), 351–376.

Ben-Tal, A. and Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research*, 23(4), 769–805.

Ben-Tal, A. and Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1), 1–13.

Bertsimas, D. and Hertog, D.D. (2022). *Robust and adaptive optimization*. Dynamic Ideas LLC.

Bitlislioğlu, A., Gorecki, T.T., and Jones, C.N. (2017). Robust tracking commitment. *IEEE Transactions on Automatic Control*, 62(9), 4451–4466.

Chisci, L., Rossiter, J.A., and Zappa, G. (2001). Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37(7), 1019–1028.

Fabietti, L., Gorecki, T.T., Qureshi, F.A., Bitlislioğlu, A., Lymperopoulos, I., and Jones, C.N. (2016). Experimental implementation of frequency regulation services using commercial buildings. *IEEE Transactions on Smart Grid*, 9(3), 1657–1666.

Goulart, P.J. and Kerrigan, E.C. (2006). A method for robust receding horizon output feedback control of constrained systems. In *Proceedings of the 45th IEEE Conference on Decision and Control (CDC)*, 5471–5476. IEEE.

Goulart, P.J. and Kerrigan, E.C. (2007). Output feedback receding horizon control of constrained systems. *International Journal of Control*, 80(1), 8–20.

Goulart, P.J., Kerrigan, E.C., and Maciejowski, J.M. (2006). Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4), 523–533.

Kim, Y., Zhang, X., Guanetti, J., and Borrelli, F. (2018). Robust model predictive control with adjustable uncertainty sets. In *2018 IEEE Conference on Decision and Control (CDC)*, 5176–5181. IEEE.

Langson, W., Chryssochoos, I., Raković, S., and Mayne, D.Q. (2004). Robust model predictive control using tubes. *Automatica*, 40(1), 125–133.

Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Scokaert, P.O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.

Mayne, D.Q., Seron, M.M., and Raković, S. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2), 219–224.

Mueller, F.L., Woerner, S., and Lygeros, J. (2019). Unlocking the potential of flexible energy resources to help balance the power grid. *IEEE Transactions on Smart Grid*, 10(5), 5212–5222.

Raghuraman, V. and Koeln, J.P. (2021). Tube-based robust MPC with adjustable uncertainty sets using zonotopes. In *2021 American Control Conference (ACC)*, 462–469. IEEE.

Rao, C.V., Rawlings, J.B., and Mayne, D.Q. (2003). Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. *IEEE transactions on automatic control*, 48(2), 246–258.

Richards, A. and How, J. (2005). Robust model predictive control with imperfect information. In *Proceedings of the 2005, American Control Conference (ACC)*, 268–273. IEEE.

Rouchier, S. (2022). Building energy statistical modelling. URL `https://buildingenergygeeks.org/index.html`.

Sieber, J., Bennani, S., and Zeilinger, M.N. (2021). A system level approach to tube-based model predictive control. *IEEE Control Systems Letters*, 6, 776–781.

Skaf, J. and Boyd, S.P. (2010). Design of affine controllers via convex optimization. *IEEE Transactions on Automatic Control*, 55(11), 2476–2487.

Vrettos, E., Oldewurtel, F., and Andersson, G. (2016). Robust energy-constrained frequency reserves from aggregations of commercial buildings. *IEEE Transactions on Power Systems*, 31(6), 4272–4285.

Zhang, X., Kamgarpour, M., Georghiou, A., Goulart, P., and Lygeros, J. (2017). Robust optimal control with adjustable uncertainty sets. *Automatica*, 75, 249–259.

Appendix A

Here we show that under a mild condition (the observer gain matrix L_o is full column rank), our considered control policy (8) subsumes the affine purified output feedback policy considered in Goulart and Kerrigan (2007) (Eq. (19)). The affine purified output feedback policy designed in Goulart and Kerrigan (2007) is

$$u_k^{\mathrm{of}} = \sum_{i=0}^{k-1} M_{k,i}(y_i - C\hat{x}_i) + v_k \tag{A.1}$$

where $y_i \in \mathbb{R}^{n_y}$ is the outputs, $C \in \mathbb{R}^{n_y \times n_x}$ is the output matrix, and $\hat{x}_i$ is the estimated states. The compacted form of u_k^{of} over the N-step prediction horizon is

$$\mathbf{u^{of}} = \mathbf{M}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{v} \tag{A.2}$$

where $\mathbf{M} \in \mathbb{R}^{Nn_u \times Nn_y}$ and $\mathbf{v} \in \mathbb{R}^{Nn_u}$ are defined as

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \ \mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}.$$

For the control policy (A.2), it can be rewritten as

$$\mathbf{u^{of}} = \mathbf{M}(\mathbf{Ce} + \boldsymbol{\eta}) + \mathbf{v} \tag{A.3}$$

where $\mathbf{e} \in \mathbb{R}^{(N+1)n_x}$ is the estimation error vector of states, $\boldsymbol{\eta} = [\eta_0, \eta_1, \cdots, \eta_{N-1}] \in \mathbb{R}^{Nn_y}$ is the output measurement error vector. According to the results in Goulart and Kerrigan (2007), for the estimated state $\hat{\mathbf{x}}$, we have

$$\hat{\mathbf{x}} = \mathbf{A}\hat{x}_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathcal{L}(\mathbf{Ce} + \boldsymbol{\eta}) \tag{A.4}$$

which further leads to

$$\mathbf{E}\mathcal{L}(\mathbf{Ce} + \boldsymbol{\eta}) = \hat{\mathbf{x}} - \mathbf{A}\hat{x}_0 - \mathbf{B}\mathbf{u} \tag{A.5}$$

where $\mathbf{E}$ is defined as in (20) and $\mathcal{L} = \mathbf{I} \otimes L_o$ with $L_o \in \mathbb{R}^{n_x \times n_y}$ being the observer gain matrix. See Goulart and Kerrigan (2007) for more details.

Assuming that the observer gain matrix L_o is full column rank, then $\mathbf{E}\mathcal{L}$ is also full row rank, and there exists a matrix $(\mathbf{E}\mathcal{L})^* = \mathcal{L}^*\mathbf{E}^* = (I \otimes L_o^*)\mathbf{E}^*$ such that $(\mathbf{E}\mathcal{L})^*\mathbf{E}\mathcal{L} = \mathbf{I}$. Consequently, we can show that

$$\mathbf{Ce} + \boldsymbol{\eta} = (\mathbf{E}\mathcal{L})^*(\hat{\mathbf{x}} - \mathbf{A}\hat{x}_0 - \mathbf{B}\mathbf{u}). \tag{A.6}$$

Substituting (A.6) into (A.3) leads to

$$\mathbf{u^{of}} = \mathbf{M}(\mathbf{E}\mathcal{L})^*(\hat{\mathbf{x}} - \mathbf{A}\hat{x}_0 - \mathbf{B}\mathbf{u^{of}}) + \mathbf{v} \tag{A.7}$$

from which we obtain

$$\begin{aligned} \mathbf{u^{of}} = {} & (\mathbf{I} + \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B})^{-1}\mathbf{M}(\mathbf{E}\mathcal{L})^*\hat{\mathbf{x}} + \\ & (\mathbf{I} + \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B})^{-1}(\mathbf{v} - \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{A}\hat{x}_0). \end{aligned} \tag{A.8}$$

Notice that $(\mathbf{I} + \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B})^{-1}$ is well-defined since

$$\mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B} = \mathbf{M}\mathcal{L}^*\mathbf{E}^*\mathbf{E}I \otimes B = \mathbf{M}\mathcal{L}^* I \otimes B$$

is strictly block lower triangular. Then, it becomes clear that there exists an equivalent affine policy of estimated states $\mathbf{u^{sf}} = \mathbf{L}\hat{\mathbf{x}} + \mathbf{g}$ where

$$\mathbf{L} := (\mathbf{I} + \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B})^{-1}\,\mathbf{M}(\mathbf{E}\mathcal{L})^* \tag{A.9a}$$

$$\mathbf{g} := (\mathbf{I} + \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{B})^{-1}\,(\mathbf{v} - \mathbf{M}(\mathbf{E}\mathcal{L})^*\mathbf{A}\hat{x}_0)\,. \tag{A.9b}$$

It can be verified that the control parameters $(\mathbf{L}, \mathbf{g})$ defined in (A.9) satisfy the structure in (9). This completes the proof. □